

Emergency Entry with One Control Torque: Non-Axisymmetric Diagonal Inertia Matrix

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I. Introduction

In Ref. 1, it was presented a method, primarily conceived as an emergency backup system, that addressed the problem of a space capsule that needed to execute a safe atmospheric entry from an arbitrary initial attitude and angular rate in the absence of nominal control capability. The proposed concept permits the arrest of a tumbling motion, orientation to the heat shield forward position and the attainment of a ballistic roll rate of a rigid spacecraft with the use of control in one axis only. To show the feasibility of such concept, the technique of single input single output (SISO) feedback linearization using the Lie derivative method² was employed and the problem was solved for different number of jets and for different configurations of the inertia matrix: the axisymmetric inertia matrix ($I_{xx} > I_{yy} = I_{zz}$), a partially complete inertia matrix with $I_{xx} > I_{yy} > I_{zz}$, $I_{xz} \neq 0$ and a realistic complete inertia matrix with $I_{xx} > I_{yy} > I_{zz}$, $I_{ij} \neq 0$. The closed-loop stability of the proposed non-linear control on the total angle of attack, θ , was analyzed through the zero dynamics of the internal dynamics³ for the case where the inertia matrix is axisymmetric ($I_{xx} > I_{yy} = I_{zz}$). This note focuses on the problem of the diagonal non-axisymmetric inertia matrix ($I_{xx} > I_{yy} > I_{zz}$), which is half way between the axisymmetric and the partially complete inertia matrices. In this note, the control law for this type of inertia matrix will be determined and its closed-loop stability will be analyzed using the same methods that were used in Ref. 1. In particular, it will be proven that the control system is stable in closed-loop when the actuators only provide a roll torque.

II. System Equations

The equations describing the control problem are those of a rotating rigid body with extra terms describing the effect of the control torques. They therefore consist of kinematic equations relating the angular position with the angular velocity, and dynamic equations describing the evolution of angular velocity.

A. Dynamic Equations

The dynamics of the rotational motion of a rigid body are described by the Euler's equations. Let I be the inertia matrix of the spacecraft, and let $\vec{\omega}$ denote the angular velocity vector with components along a body-fixed reference frame located at the center of gravity (c.g.) and aligned along the principal axes of the spacecraft. The dynamic equations are presented in Eq. (1), where $\vec{k}u$ denotes the control torque vector.

$$I\dot{\vec{\omega}} = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix} I\vec{\omega} + \vec{k}u = \omega^\times I\vec{\omega} + \vec{k}u \quad (1)$$

B. Kinematic Equations

The kinematic equations relate the components of the angular velocity vector with the rates of a set of parameters that describe the relative orientation of two reference frames: an inertial frame and the body-

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fixed frame. In the formulation of this problem, the inertial frame $O\hat{X}_V, \hat{Y}_V, \hat{Z}_V$ is associated to the velocity vector. It is defined with its origin located at the c.g. of the spacecraft, with the \hat{X}_V axis pointed along the velocity vector, the \hat{Y}_V axis pointed along the angular momentum of the trajectory, and \hat{Z}_V axis pointed along $\hat{Y}_V \times \hat{X}_V$. The body-fixed frame $O\hat{x}, \hat{y}, \hat{z}$ is defined by the body axes. In this case, when the rotation angles are zero, the \hat{x} axis goes along the longitudinal axis, aligned with \hat{X}_V , being positive toward the heat shield, \hat{y} is aligned with $-\hat{Y}_V$ and \hat{z} with \hat{Z}_V . For convenience, we will consider a 1-3-1 sequence of rotations which is a type 2 Euler sequence. The rotations involved are φ about \hat{X}_V , θ about \hat{Z}'_V and ψ about \hat{x} , where \hat{Z}'_V is the resulting \hat{Z}_V after the first rotation. This selection results in the kinematic equations, Eq. (2)

$$\begin{aligned}\dot{\varphi} &= (-\omega_y \cos \psi + \omega_z \sin \psi) \csc \theta \\ \dot{\theta} &= \omega_y \sin \psi + \omega_z \cos \psi \\ \dot{\psi} &= \omega_x + (\omega_y \cos \psi - \omega_z \sin \psi) \cot \theta\end{aligned}\tag{2}$$

C. Equations of Motion

Combining Eq. (1) and Eq. (2) yield the attitude control problem. We can see that θ , the variable that we want to control first, only depends on ω_y , ω_z and ψ in the kinematic equations, therefore the equation for $\dot{\varphi}$ can be ignored. This makes sense since θ , as well as its angular components, angle of attack and sideslip angle, are invariant to a rotation around the velocity vector. Hence the equations of motion can be reduced to

$$\begin{aligned}\ddot{\omega} &= I^{-1}\omega^\times I\vec{\omega} + I^{-1}\vec{k}u \\ \dot{\theta} &= \omega_y \sin \psi + \omega_z \cos \psi \\ \dot{\psi} &= \omega_x + (\omega_y \cos \psi - \omega_z \sin \psi) \cot \theta\end{aligned}\tag{3}$$

III. Nonlinear SISO System

The system in Eq. (3) can be written in the form of a SISO system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\quad \text{like} \quad \begin{aligned}x &= (\omega_x \quad \omega_y \quad \omega_z \quad \theta \quad \psi)^T \\ f(x) &= \begin{pmatrix} I^{-1}\omega^\times I\omega \\ \omega_y \sin \psi + \omega_z \cos \psi \\ \omega_x + (\omega_y \cos \psi - \omega_z \sin \psi) \cot \theta \end{pmatrix} \\ g(x) &= (I^{-1}\vec{k} \quad 0 \quad 0)^T, \quad h(x) = \theta - \theta_d\end{aligned}\tag{4}$$

with $x \in \mathbb{R}^n$, where n is the system order (5 in our case), with f and g smooth vector fields on \mathbb{R}^n and h a smooth nonlinear function, and where θ_d is the desired, or targeted θ . A large class of SISO nonlinear systems can be made to have linear input-output behavior through a choice of nonlinear state feedback control law that is given with generality by

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)} \left(-L_f^\gamma h(x) + v \right)$$

where $L_f h(x)$ and $L_g h(x)$ stand for the Lie derivatives of h with respect to f and g respectively, γ is the relative degree of the system and the input v is a scalar that results from the product of the feedback gain vector times the error vector. This control law yields the γ -th order linear system from input v to input y : $\frac{d^\gamma y}{dt^\gamma} = v$, where γ is the smallest integer for which $L_g L_f^i h(x) \equiv 0$, $i = 0, \dots, \gamma-2$. It is easier to understand γ as the smallest integer for which the control signal appears in $d^\gamma y/dt^\gamma$ for the first time.

By inspection of Eq. (4) we can see that the relative degree γ will depend on the specific form of the product $I^{-1}\vec{k}$. Therefore, both, the inertia matrix and the geometric layout of the jets are the two factors that affect the relative degree of the system. Note that γ can only take the values 2 or 3: it cannot be 1 since the first derivative of the output, $\dot{\theta}$, does never contain the control; it is 2 if the vector $I^{-1}\vec{k}$ has either its 2nd or 3rd component different than zero; and γ would be 3 if the vector $I^{-1}\vec{k}$ only had its 1st component different than zero. A γ higher than 3 is not possible.

The relative degree, and hence the product $I^{-1}\vec{k}$, has important implications in the stability characteristics of the system under control since the dimension of its internal dynamics is given precisely by $n - \gamma$. When γ is defined and smaller than n , the nonlinear system in Eq. (4) can be transformed into a so called normal form, which shall allow us to take a formal look at the stability of the system through the notions of internal dynamics and zero dynamics. The normal states become $\Phi(x) = (\xi_1 \cdots \xi_\gamma \eta_1 \cdots \eta_{n-\gamma})^T$ and the normal form of the system can be written as

$$\begin{aligned} \dot{\xi}_1 &= L_f h(x) = \xi_2 & \dot{\eta}_1 &= L_f \eta_1 = q_1(\xi, \eta) \\ \dot{\xi}_2 &= L_f^2 h(x) = \xi_3 & \dot{\eta}_2 &= L_f \eta_2 = q_2(\xi, \eta) \\ \vdots & & \vdots & \\ \dot{\xi}_\gamma &= L_f^\gamma h(x) + L_g L_f^{\gamma-1} h(x) u & \dot{\eta}_{n-\gamma} &= L_f \eta_{n-\gamma} = q_{n-\gamma}(\xi, \eta) \\ & & h(x) &= \xi_1 \end{aligned} \quad (5)$$

where the vector field η is a solution of the set of partial differential equations

$$\nabla \eta_j(x) g(x) = 0 \quad 1 \leq j \leq n - \gamma \quad (6)$$

To show that the nonlinear system in Eq. (4) can indeed be transformed into the normal form of Eq. (5), we have to show that we can construct a local diffeomorphism $\Phi(x)$ such that Eq. (5) is verified. To show that $\Phi(x)$ is a diffeomorphism, it suffices to show that its Jacobian $\partial\Phi/\partial x$ is invertible, i.e., that the gradients $\nabla \xi_i$ and $\nabla \eta_j$ are linearly independent ($\text{span}(\partial\Phi(x)/\partial x) \in \mathbb{R}^5$).

The internal dynamics associated with the input-output linearization correspond to the last $n - \gamma$ equations of the normal form and they constitute the unobservable part. However, the control must account for the stability of the whole dynamics and, therefore, the system will be stable if the internal dynamics remain bounded. To assess the stability of the internal dynamics we will use the zero dynamics of the system, which is defined as the internal dynamics of the system when its output is kept at zero by the input. As mentioned in Ref. 3, the zero dynamics is an intrinsic feature of a nonlinear system that does not depend on the choice of the control law or the desired trajectories.

The constraint that the output h is identically zero in zero dynamics implies that all of its time derivatives are zero. Thus, the corresponding internal dynamics of the system, or zero dynamics, describes a motion restricted to the $n - \gamma$ dimensional manifold defined by $\xi = 0$. Also, in order for the system to be in zero dynamics, the input control u must be such that h stays at zero. This means that in zero dynamics the normal form can be written as $(\dot{\xi}_1 = 0, \dots, \dot{\xi}_\gamma = 0, \dot{\eta}_1 = q_1(0, \eta), \dots, \dot{\eta}_{n-\gamma} = q_{n-\gamma}(0, \eta), h = 0)$

The analysis of the zero dynamics of the internal dynamics will be carried out for the control on θ in the axis-symmetric inertia case only. The resulting expressions in the intermediate and realistic inertias are practically intractable. Nonetheless, we will see that useful conclusions can be from the axis-symmetric case can be extended to the other more realistic cases.

IV. Closed-loop Stability with Roll Torque

In the case in which we have a diagonal non-axisymmetric matrix of inertia and in which only a roll torque is provided, the equations of motion become

$$\begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} a\omega_x\omega_z \\ b\omega_x\omega_z \\ c\omega_x\omega_y \\ \omega_y \sin \psi + \omega_z \cos \psi \\ \omega_x + (\omega_y \cos \psi - \omega_z \sin \psi) \cot \theta \end{pmatrix} + \begin{pmatrix} I_{xx}^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u \quad (7)$$

where $a = (I_{yy} - I_{zz})/I_{xx}$, $b = (I_{zz} - I_{xx})/I_{yy}$ and $c = (I_{xx} - I_{yy})/I_{zz}$.

The relative degree of the system, γ , is 3 and, thus, the control law will be given by $u = (v - L_f^3 \theta)/(L_g L_f^2 \theta)$, which is shown in Appendix A.

The external part of the normal system is given by

$$\begin{aligned}\xi_1 &= \Phi_1 = h(x) = \theta - \theta_d \\ \xi_2 &= \Phi_2 = L_f h(x) = \dot{\theta} \\ \xi_3 &= \Phi_3 = L_f^2 h(x) = \ddot{\theta}\end{aligned}$$

where $L_f h(x)$ stands for the Lie derivative of h with respect to f .

The internal part of the normal system will have 2 components and must satisfy $\nabla \eta_j(x)g(x) = 0$

$$\begin{aligned}\frac{\partial \eta_1}{\partial x} g(x) &= \frac{\partial \eta_1}{\partial \omega_x} I_{xx}^{-1} = 0 \\ \frac{\partial \eta_2}{\partial x} g(x) &= \frac{\partial \eta_2}{\partial \omega_x} I_{xx}^{-1} = 0\end{aligned}$$

The solution of the internal system must result in $\Phi(x) = (\xi_1 \ \xi_2 \ \xi_3 \ \eta_1 \ \eta_2)^T$ being a diffeomorphism. One solution that satisfies this requirement is given by $\eta_1 = \omega_y$, $\eta_2 = \omega_z$. Therefore, the internal system becomes

$$\begin{aligned}\dot{\eta}_1 &= \dot{\omega}_y = b\omega_x\omega_z \\ \dot{\eta}_2 &= \dot{\omega}_z = c\omega_x\omega_y\end{aligned}\tag{8}$$

In the axisymmetric case, it could be demonstrated that ω_x was constant in zero dynamics. However, that is not necessarily the case now. To check whether the system in Eq. 8 is stable, $\dot{\omega}_y$ is divided by $\dot{\omega}_z$, resulting in

$$\frac{\dot{\omega}_y}{\dot{\omega}_z} = \frac{b\omega_z}{c\omega_y}\tag{9}$$

Operating on Eq. 9 and integrating results in

$$\int \omega_y d\omega_y - \frac{b}{c} \int \omega_z d\omega_z = 0 \Rightarrow \omega_y^2 - \frac{b}{c} \omega_z^2 = C\tag{10}$$

where C is the integration constant which includes the factor 1/2 coming from the integration. Eq. 10 can be rewritten as

$$\frac{\omega_y^2}{C} + \frac{\omega_z^2}{C(-c/b)} = 1\tag{11}$$

which is the standard equation of an ellipse. Note that in Eq. 11 the fraction c/b is negative because $I_{xx} > I_{yy} > I_{zz}$.

This mathematical development proves that the internal system in Eq. 8 is bounded and thus, that the closed-loop control system can be said to be stable.

A. Test Case

In the case of an axisymmetric matrix of inertia the transversal rate $\omega_T = \omega_{T0} = (\omega_{y0}^2 + \omega_{z0}^2)^{1/2}$ was constant throughout the control process. Now, this condition does not hold. Nevertheless, the settling time will be determined following the logic that was used in the axisymmetric case presented in Ref. 1. A settling time $t_s = (2\pi/\omega_{T0})\delta$ is selected, where δ is a factor that represents some percentage of added or subtracted time. In this case, $\delta = 1.25$, which represents a 25% time addition. Nevertheless, for all practical purposes, the important time is the one that marks the moment where θ becomes confined below the side wall angle, which is always smaller than t_s .

For the design of the linear controller, the feedback gain vector \vec{K} is calculated using a pole placement technique. It is selected such that a desired set of closed-loop pole locations \vec{p} is matched. The location of the poles was chosen such that there was no overshoot in θ , at least when the control signal is not saturated. The location of the poles will vary with ω_{T0} in the following manner:

$$p = (-1 \quad -5 \quad -7) 5/t_s\tag{12}$$

Figure 1 presents the results for the case in which the initial rates are at the edge of a typical rate deadband limits. The moments of inertia are given by $I_{xx} = 5900 \text{ kg m}^2$, $I_{yy} = 5100 \text{ kg m}^2$ and $I_{zz} = 4700$

kg m². This case has the following initial conditions: $\omega_{x0} = 2$ deg/sec, $\omega_{y0} = -2$ deg/sec, $\omega_{z0} = -2$ deg/sec, $\theta_0 = 150$ deg, $\psi_0 = 120$ deg, $\theta_d = 9$ deg, $t_s = 159.1$ sec. Figure 1 is divided in two parts. The left side presents the locus of the positive direction of the spacecraft's longitudinal axis (+x axis) in the projection of the two dimensional attitude sphere in terms of the angles of attack and sideslip. Each of the contours represents an iso- θ trajectory spaced every ten degrees around the velocity vector, located at the center of the diagram. A spacecraft exactly flying heat shield forward would show its locus right at (0,0), whereas a spacecraft exactly flying apex forward would show its locus at (0,±180). The ring in the center represents a typical side wall angle of 32.5 deg, therefore, as a first approximation, to have the capsule protected from the heat of the entry means that θ should remain confined within that ring. The right side of the chart shows the time histories of the control signal, the total angle of attack and the roll rate. The control signal is expressed as a percentage of the maximum acceleration that the system can generate; the plot showing θ also indicates the side wall angle as a dashed horizontal line and the settling time t_s ; and the roll rate plot also shows the closest ballistic roll rate value $|p_b|$ or $-|p_b|$. The case considered in this note is that of a ballistic capsule because there are no products of inertia and, thus, the center of gravity is contained in the axis of symmetry of the spacecraft. In any case, the roll rate is shown because it is a byproduct of this type of control acting on the roll axis. Should the resulting roll rate be so high that it violates the medical constraints on angular rates or a possible constraint on the deploy of the landing system, the roll rate could be controlled in the presence of atmosphere as it is explained in chapter VII in Ref 1.

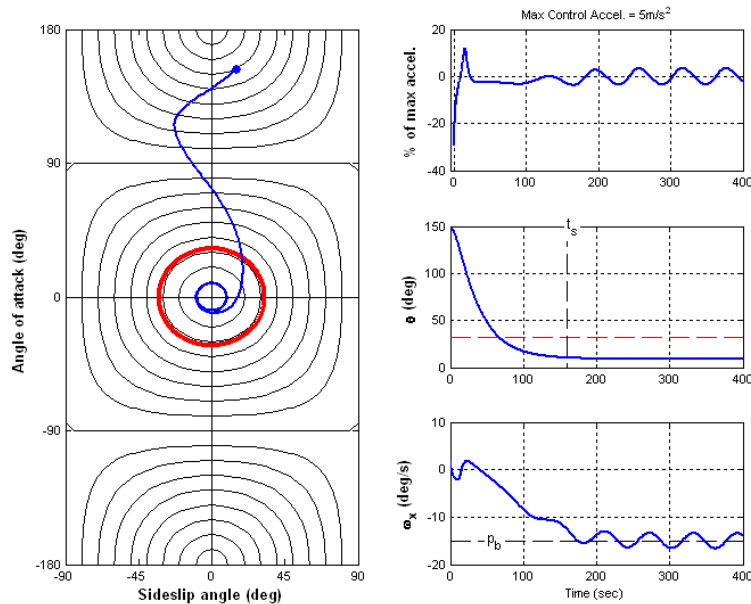


Figure 1. Test case results.

It can be noted in Figure 1 that the control signal becomes periodic in steady state. Nevertheless, the control can be turned off after $t = t_s$, a point where the error signal is close to zero. That condition ensures the total angle of attack confinement and a ballistic entry. Figure 2 shows the results when the control signal is turned off at $t = 175.0$ sec.

V. Conclusion

It has been shown that for the diagonal non-axisymmetric matrix of inertia with relative degree 3, there exists a feedback control law that can transfer the system to a desired total angle of attack with $\dot{\theta} = \ddot{\theta} = 0$ while the rest of the state variables remain bounded. A feedback control law based on Feedback Linearization that satisfies the above condition has been formulated. The controller's gains were designed to depend on the initial conditions and they were generated following a standard pole placement methodology.

Also, for the diagonal non-axisymmetric inertia matrix, the closed loop stability of the system when controlling θ with roll torque was demonstrated to be globally stable.

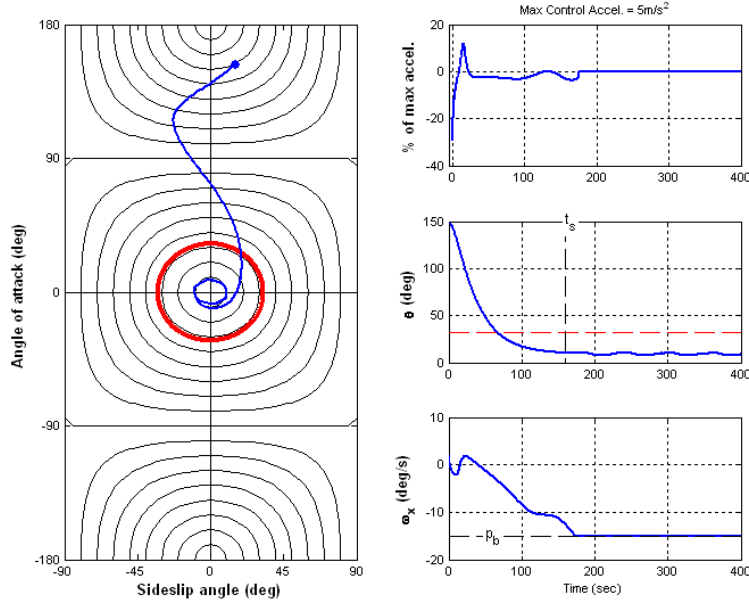


Figure 2. Test case results. Control turned off at $t = 175.0$ sec.

Appendix A - Derivation of the Control Law

The control law u is given by

$$u = (v - L_f^3 \theta) / (L_g L_f^2 \theta) \quad (13)$$

where

$$\begin{aligned} L_f \theta &= (\partial \theta / \partial x) f(x) = \omega_y \sin \psi + \omega_z \cos \psi \\ L_f^2 \theta &= (\partial L_f \theta / \partial x) f(x) = \omega_x [\omega_y \cos \psi (c+1) + \omega_z \sin \psi (b-1)] + (\omega_y \cos \psi + \omega_z \sin \psi)^2 \cot \theta \\ L_f^3 \theta &= (\partial L_f^2 \theta / \partial x) f(x) = [\omega_y \cos \psi (c+1) + \omega_z \sin \psi (b-1)] \dot{\omega}_x \\ &\quad + [\omega_x \cos \psi (c+1) + 2(\dot{\psi} - \omega_x) \cos \psi] \dot{\omega}_y \\ &\quad + [\omega_x \sin \psi (b-1) - 2(\dot{\psi} - \omega_x) \sin \psi] \dot{\omega}_z \\ &\quad - \dot{\varphi}^2 \dot{\theta} \\ &\quad + \left[[\omega_z \cos \psi (b-1) - \omega_y \sin \psi (c+1)] - 2(\dot{\psi} - \omega_x) \dot{\theta} \right] \dot{\psi} \\ L_g L_f^2 \theta &= (\partial L_f^2 \theta / \partial x) g(x) = I_{xx}^{-1} [\omega_y \cos \psi (c+1) + \omega_z \sin \psi (b-1)] \end{aligned}$$

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